Summary of rules for uncertainty propagation

This document is meant to be a concise guide on the propagation of uncertainties in calculations – an important skill for your PHY 151/152 Practicals and beyond. For further information, students are referred to the *Uncertainty* modules of the Practicals website, or to a textbook on measurements and uncertainties[1].

1 Generalities

Throughout this document, the uncertainty on a quantity x is denoted by u_x, with $u_x > 0$.

Uncertainty Propagation First, a word on what is meant by *uncertainty propagation*. Any measured quantity has an uncertainty associated with it[2]; indeed, it is fair to say that a measurement without an uncertainty has no meaning. This uncertainty would have been estimated or calculated by some means; please refer to other references for details on how uncertainties are estimated and calculated.

Typically, one would like to do something of interest with this measurement, like calculating secondary quantity to better interpret the experiment or to compare with expectations. Clearly, such a secondary quantity should also carry an uncertainty, like the primary quantity on which it depends. The set of principles for determining the uncertainties on such secondary, calculated quantities are referred to as *uncertainty propagation*.

Addition in Quadrature The uncertainty propagation rules described herein use *addition in quadrature* when combining uncertainties from several measurements. This prescription, which may seem surprising at first, is in fact the result of a proper, probabilistic treatment of uncertainties: remember that the uncertainty on a measurement is the estimated standard deviation of the underlying probability distribution, assumed to be a Gaussian. For this reason, a secondary quantity that is calculated from a measured value should also be interpreted as a Gaussian-distributed value.

It turns out that when Gaussian-distributed values are combined, their standard deviations combine in quadrature: this fact underpins the formulae presented in this document. Importantly, this holds true *provided the different quantities are independent*, probabilistically speaking. Otherwise, these uncertainty propagation rules are not valid.

Reporting Uncertainties We usually report uncertainties with *one* significant digit; for example, (5.9 ± 0.4) cm, where 5.9 cm is called the *measurand* and 0.4 cm is the uncertainty. Then, the number of significant digits in the measurand is set by the decimal place of the uncertainty. For instance, if a calculation yields the quantity (13.367 ± 0.812) s, then it is reported as (13.4 ± 0.8) s.

This rule is not ironclad, however: in cases where rounding the uncertainty to one significant digit would change its value by a large percentage, it may be preferable to leave two significant digits. In that case, the number of significant digits in the measurand is set by the *last* decimal place of the uncertainty. For example, it would be reasonable to report (13.345 ± 0.149) kg as (13.35 ± 0.15) kg, rather than as (13.4 ± 0.1) kg.

In all cases, it is best to not round off values of both measurand and uncertainty until the very end of a calculation. This could be achieved, for instance, by storing them in your calculator's memory.

2 Addition and subtraction rule

If c is the sum or difference of a and b, that is to say

$$c = a \pm b, \tag{1}$$

then the uncertainty u_c on c is given by the relation

$$u_c{}^2 = u_a{}^2 + u_b{}^2. \tag{2a}$$

It is then trivial to solve for u_c and obtain

$$u_c = \sqrt{u_a{}^2 + u_b{}^2}. \tag{2b}$$

This rule can be iterated to arrive at the following result for a string of additions and/or subtractions:

$$c = a_1 + a_2 + \cdots - b_1 - b_2 - \cdots \tag{3}$$

$$u_c{}^2 = u_{a_1}{}^2 + u_{a_2}{}^2 + \cdots + u_{b_1}{}^2 + u_{b_2}{}^2 + \cdots. \tag{4}$$

3 Multiplication and division rule

If c is the product or the quotient of a with b, that is to say

$$c = ab \tag{5a}$$

or

$$c = \frac{a}{b}, \tag{5b}$$

[1] For instance, *Measurements and their Uncertainties: A Practical Guide to Modern Error Analysis* by Ifan Hughes and Thomas Hase (2010).

[2] Well, almost any – see the remark in section 6.

then the squared *relative* uncertainty[3] of c is given by the *relative* uncertainties of a and b added in quadrature:

$$\left(\frac{u_c}{c}\right)^2 = \left(\frac{u_a}{a}\right)^2 + \left(\frac{u_b}{b}\right)^2. \tag{6a}$$

It is then a trivial matter to solve for the uncertainty u_c:

$$u_c = |c|\sqrt{\left(\frac{u_a}{a}\right)^2 + \left(\frac{u_b}{b}\right)^2}. \tag{6b}$$

Once again, the rule is easily iterated in the case of repeated multiplication and division:

$$c = \frac{a_1 a_2 \dots}{b_1 b_2 \dots} \tag{7}$$

$$\begin{aligned}\left(\frac{u_c}{c}\right)^2 = \left(\frac{u_{a_1}}{a_1}\right)^2 + \left(\frac{u_{a_2}}{a_2}\right)^2 + \cdots \\ + \left(\frac{u_{b_1}}{b_1}\right)^2 + \left(\frac{u_{b_2}}{b_2}\right)^2 + \cdots.\end{aligned} \tag{8}$$

4 General case: the derivative rule

4.1 Functions of a single variable

The derivative rule provides a way to propagate uncertainty in the case of an arbitrary function. The idea is that for small u_x, the ratio u_y/u_x is approximately the same as the slope $\mathrm{d}y/\mathrm{d}x$, which we can use to find u_y.

Let $y = f(x)$. Then, the uncertainty on y is

$$u_y = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| u_x. \tag{9}$$

This rule has an intuitive interpretation: the uncertainty interval on the y axis is found by simply scaling that on the x axis by the slope of f near x (presuming the linear approximation of f near x is good on that uncertainty interval).

4.2 Functions of multiple variables

Let $y = f(x_1,\dots,x_n)$, where n is a natural number. Then,

$${u_y}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 {u_{x_1}}^2 + \cdots + \left(\frac{\partial f}{\partial x_2}\right)^2 {u_{x_n}}^2. \tag{10}$$

It is easy to show that the rules from sections 2 and 3 are special cases of this one.

5 Examples

5.1 Addition and subtraction

Suppose we seek the vertical displacement of a projectile fired straight up, from its point of departure to its maximum height. The ball was put in motion at a height of $y_\mathrm{i} = (14.2 \pm 0.5)\,\mathrm{cm}$. Its highest altitude, measured in flight (with less precision), was $y_\mathrm{f} = (43 \pm 2)\,\mathrm{cm}$. Then,

$$\begin{aligned}h &= y_\mathrm{f} - y_\mathrm{i} \\ &= 43\,\mathrm{cm} - 14.2\,\mathrm{cm} \\ &= 28.8\,\mathrm{cm}.\end{aligned} \tag{11}$$

The uncertaity on h is given by

$$\begin{aligned}u_h &= \sqrt{(0.5\,\mathrm{cm})^2 + (2\,\mathrm{cm})^2} \\ &= \sqrt{4.25\,\mathrm{cm}^2} \\ &= 2.061552813\,\mathrm{cm}\end{aligned} \tag{12}$$

Hence, following the rounding rules, we would report the result as

$$h = (14 \pm 2)\,\mathrm{cm}. \tag{13}$$

This example illustrates a quality of addition in quadrature: the largest uncertainties are made even more dominant, which, in conjunction with the rounding rules, means that smaller uncertainties can often be ignored. Such cases should become recognizable with practice.

5.2 Multiplication and division

Continuing with the previous situation, suppose we now want to compute the work done on the projectile, whose mass is $m = (250 \pm 1)\,\mathrm{g}$, by the force of gravity. This is given by

$$\begin{aligned}W &= -mgh \\ &= -(0.250\,\mathrm{kg})(9.81\,\mathrm{N/kg})(0.288\,\mathrm{m}) \\ &= 0.70632\,\mathrm{J}.\end{aligned} \tag{14}$$

Neglecting any uncertainy in g, u_W is given by

$$\begin{aligned}u_W &= W\sqrt{\left(\frac{u_m}{m}\right)^2 + \left(\frac{u_h}{h}\right)^2} \\ &= 0.70632\,\mathrm{J}\sqrt{\left(\frac{1\,\mathrm{g}}{250\,\mathrm{g}}\right)^2 + \frac{4.25\,\mathrm{cm}^2}{(28.8\,\mathrm{cm})^2}} \\ &= 0.70632\,\mathrm{J}\sqrt{1.6\times10^{-5} + 5.1239\times10^{-3}} \\ &= 0.70632\,\mathrm{J}\times 0.0716933682 \\ &= 0.05063845982\,\mathrm{J}\end{aligned} \tag{15}$$

[3] We call *relative uncertainty* the ratio of the uncertainty with the measurand, i.e. u_x/x. In contrast, the quantity u_x is sometimes called the *absolute uncertainty* for clarity.

Note that in both equations 14 and 15, the unrounded values for h and u_h were used, and all intermediate values were stored in calculator memory. Hence, we find that

$$W = (0.71 \pm 0.05)\,\mathrm{J}. \tag{16}$$

In this case, we could have noticed from the start that the *relative* uncertainty on h is far greater than that on m, so the latter could have safely been ignored without affecting the final answer.

5.3 Functions of a single variable

Suppose $y = \sin\theta$, where θ was measured to be $(34.2 \pm 0.5)^\circ$. For y, we find

$$\begin{aligned} y &= \sin(34.2^\circ) \\ &= 0.562083378. \end{aligned} \tag{17}$$

According to equation 9, the uncertainty on y is given by

$$\begin{aligned} u_y &= \left|\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\right| u_\theta \\ &= |\cos\theta|\, u_\theta. \end{aligned} \tag{18}$$

This example allows us to make a crucial point on uncertainties and trigonometric functions: *since calculus is done in radians, angles must be converted to radians in the computation of uncertainties.* Hence, via the conversion $180^\circ = \pi\,\mathrm{rad}$, we find $u_\theta = 8.72664626 \times 10^{-3}\,\mathrm{rad}$, and therefore

$$\begin{aligned} u_y &= \left|\cos 34.2^\circ\right| 8.72664626 \times 10^{-3}\,\mathrm{rad} \\ &= 0.8270805743 \times 8.72664626 \times 10^{-3}\,\mathrm{rad} \\ &= 0.8270805743 \times 8.72664626 \times 10^{-3}\,\mathrm{rad} \\ &= 7.2176396 \times 10^{-3}. \end{aligned} \tag{19}$$

Note that since angles expressed in radians are technically dimensionless, we freely dropped the ‘rad’ in the last step. The final answer is reported as

$$y = 0.562 \pm 0.007. \tag{20}$$

5.4 Functions of multiple variables

Suppose we seek to calculate the number of particles which have not yet undergone some radioactive decay at time $t = (10.0 \pm 0.1)\,\mathrm{min}$, given by the formula

$$n = n_0 \mathrm{e}^{-t/\tau}. \tag{21}$$

The initial number of particles is somehow known to be $n_0 = (3.81 \pm 0.05) \times 10^{20}$, and the time constant, $\tau = (31.2 \pm 0.1)\,\mathrm{min}$. To use the rule of equation 10, we must evaluate the following derivatives[4]:

$$\frac{\partial n}{\partial n_0} = \mathrm{e}^{-t/\tau} = \frac{n}{n_0}, \tag{22a}$$

$$\frac{\partial n}{\partial t} = -\frac{n_0}{\tau}\mathrm{e}^{-t/\tau} = -\frac{n}{\tau}, \tag{22b}$$

$$\frac{\partial n}{\partial \tau} = \frac{t}{\tau^2} n_0 \mathrm{e}^{-t/\tau} = \frac{t}{\tau^2} n. \tag{22c}$$

Hence, the uncertainty on n is conveniently expressed as

$$\begin{aligned} u_n &= \sqrt{\left(\frac{n}{n_0}\right)^2 {u_{n_0}}^2 + \left(\frac{n}{\tau}\right)^2 {u_t}^2 + \left(\frac{nt}{\tau^2}\right)^2 {u_\tau}^2} \\ &= |n| \sqrt{\left(\frac{u_{n_0}}{n_0}\right)^2 + \left(\frac{u_t}{\tau}\right)^2 + \left(\frac{u_\tau t}{\tau^2}\right)^2}. \end{aligned} \tag{23}$$

Explicit substitution of numerical values is straightforward, and is left as an exercise for the reader.

6 Additional remarks

For completeness, let us note that some physical quantities don’t have uncertainties associated to them. This is the case for defined constants (such as the speed of light in a vacuum and the permeability of free space), as well as – in principle – quantities that can only take on integer values, like the number of β particles emitted from a radioactive source[5].

[4] For convenience, we re-express the derivatives in terms of the quantity n, which will have been evaluated according to equation 21 beforehand.

[5] The example of subsection 5.4, however, illustrates that the latter point may be moot when dealing with macroscoping numbers of particles, in which case experiments cannot resolve their discrete number.